The Diamond Model

References

- Acemoglu Ch. 9, Maussner/Klump 132-140.
- Diamond, Peter A., 1965, National Debt in a Neoclassical Growth Model, *American Economic Review* 55,1126-1150.

We will discuss a simplified version (plus an interesting extension).

Idea:

- Individual life is finite.
- The economy goes on forever.
- 2 distinguishable periods in life: young (1) and old (2)
- Only the young work.
- They save for old age. $\rightarrow$ This delivers an endogenous explanation of the savings rate.
- At each t: 2 generations exist simultaneously (overlap) –> OLG model.
- Time is discrete (one period = one generation).

Way of life: People are born, work and save during young adulthood and live off their savings in old age.

- every young adult supplies 1 unit labor
- no bequests, no altruism towards offsprings.

People maximize life-time utility (for simplicity we focus on log-utility):

$$U = \log(c_t) + \beta \log(c_{t+1}). \tag{1}$$

β: discount factor of future consumption c_{t+1}, $0 < \beta < 1$.

Future consumption equals current savings s_t plus interest: $c_{t+1} = (1 + r_{t+1})s_t$

Wage income can be consumed or saved:

$$w_t = c_t + s_t \qquad \Rightarrow \qquad w_t - c_t - \frac{c_{t+1}}{1 + r_{t+1}} = 0. \tag{2}$$

Lagrangian:

$$L = \log(c_t) + \beta \log(c_{t+1}) + \lambda \left[w_t - c_t - \frac{c_{t+1}}{1 + r_{t+1}} \right]$$

FOCs:

$$\frac{1}{c_t} = \lambda$$

$$\beta \frac{1}{c_{t+1}} = \frac{\lambda}{1 + r_{t+1}}$$

And thus

$$\frac{c_{t+1}}{c_t} = \beta(1 + r_{t+1}). \tag{3}$$

Insert the "Ramsey rule" (3) into the budget constraint (2):

$$w_t = c_t + \frac{1 + r_{t+1}}{1 + r_{t+1}} \beta c_t = (1 + \beta) c_t$$

and thus

$$c_t = \frac{1}{1 + \beta} w_t$$

The savings rate fulfills $c_t = (1 - s) w_t$. Thus

$$(1 - s) = \frac{1}{1 + \beta} \qquad \Rightarrow \qquad s = \frac{\beta}{1 + \beta}. \tag{4}$$

Production:

$$Y = AK^{\alpha} L^{1-\alpha} \tag{5}$$

implying

$$r + \delta = \alpha A K^{\alpha-1} L^{1-\alpha} = \alpha A k^{\alpha-1}$$
$$w = (1-\alpha) A K^{\alpha} L^{-\alpha} = (1-\alpha) A k^{\alpha}.$$

equation of motion for capital stock:

$$K_{t+1} = s \cdot w_t \cdot L_t + (1-\delta) K_t$$

Assume (for now) constant population, i.e. $L_t = L_{t+1}$. Thus

$$\frac{K_{t+1}}{L_{t+1}} = k_{t+1} = s w_t + (1-\delta) k_t \quad \Rightarrow \quad k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha) A k_t^{\alpha} + (1-\delta) k_t \tag{6}$$

Graphical equilibrium analysis:

[Diamond model: equilibrium analysis]

Conclude:

- the non-trivial equilibrium is unique
- and globally stable.

Observe:

- $\beta \downarrow \;\; \rightarrow s \downarrow \;\; \rightarrow k^* \downarrow \;\; \rightarrow y \downarrow$.
- so far everything as predicted by the Solow model, only savings are endogenously explained.

Now consider a variant of the model. 2 Changes.

(i) Subsistence consumption $\bar{c}$

- What's that? (a good question).
- Here we use the following definition. If $w \leq \bar{c}_t$
  - the people get the lowest utility possible ($-\infty$)
  - they cannot afford to save
- Recall the introduction. Think of $\bar{c} = 1$ Dollar a day.

$$U = \begin{cases} \log(c_t - \bar{c}) + \beta \log(c_{t+1}) & \text{for } c_t \geq \bar{c} \\ -\infty & \text{otherwise} \end{cases} \tag{7}$$

Budget constraint is before.

Lagrangian:

$$L = \log(c_t - \bar{c}) + \beta \log(c_{t+1}) + \lambda \left[w_t - c_t - \frac{c_{t+1}}{1 + r_{t+1}} \right]$$

FOCs:

$$\frac{1}{c_t - \bar{c}} = \lambda$$
$$\beta \frac{1}{c_{t+1}} = \frac{\lambda}{1 + r_{t+1}}$$

if $w_t > \bar{c}$. Thus

$$c_{t+1} = \begin{cases} \beta(1 + r_{t+1})(c_t - \bar{c}) & \text{for } w_t > \bar{c} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Insert the interior solution into the budget constraint (2):

$$w_t = c_t + \frac{1}{1 + r_{t+1}} \beta(1 + r_{t+1}) (c_t - \bar{c})$$

implying

$$w_t = c_t + \beta c_t - \beta \bar{c} = (1 + \beta) c_t - \beta \bar{c}$$

for $w_t > \bar{c}$. And thus...

$$c_t = \frac{1}{1+\beta} w_t + \frac{\beta}{1+\beta} \bar{c} = \left(\frac{1}{1+\beta} + \frac{\beta}{1+\beta} \frac{\bar{c}}{w_t} \right) \cdot w_t = (1 - s_t) w_t .$$

Implying the savings rate:

$$s_t = 1 - \frac{1}{1+\beta} - \frac{\beta}{1+\beta} \frac{\bar{c}}{w_t} = \frac{\beta}{1+\beta} \left(1 - \frac{\bar{c}}{w_t} \right) \tag{9}$$

Observe:

- The savings rate is no longer constant.
- It is zero for wage income below $\bar{c}$.
- It is monotonously increasing in income afterwards.
- It approaches a maximum $\beta/(1+\beta)$ for $w \to \infty$.
- This is (stylized) what we observe.
- The neoclassical model gains in plausibility.

[Insert: savings with subsistence consumption]

Yet, this is just the first feature of the model's extension. We get also a poverty trap.

(ii) Consider production:

$$Y = A(K + X)^{\alpha} L^{1-\alpha} \tag{10}$$

- Here, X is arable land.
- Land is assumed to be constant and owned by a negligible fraction of the population (the landed aristocracy).
- Note: there will be $Y > 0$ without K.

Implied wages:

$$w = (1-\alpha)A(K + X)^{\alpha} L^{-\alpha} = (1-\alpha)A\left(k + \frac{X}{L}\right)^{\alpha} \tag{11}$$

Insert savings (9) and wages (11) into the equation of motion

$$K_{t+1} = s_t w_t L_t + (1-\delta)K_t \qquad \Rightarrow \qquad k_{t+1} = \frac{\beta}{1+\beta}\left(1 - \frac{\bar{c}}{w_t}\right) w_t + (1-\delta)k_t$$

for $w_t > \bar{c}$. And thus altogether:

$$k_{t+1} = \max\left(0, f(k_t)\right), \quad \text{where}$$
$$f(k_t) = \frac{\beta}{1+\beta}\left[(1-\alpha)A\left(k_t + \frac{X}{L}\right)^{\alpha} - \bar{c}\right] + (1-\delta)k_t \tag{12}$$

- Note that that there exists a k_0 with $f'(k_0) > 0$ and $f(k_0) = 0$.
- Thus there exists a locally stable equilibrium at $k^* = 0$ with

$$c_t = w_t = (1-\alpha)A\left(\frac{L}{X}\right)^{\alpha}. \tag{13}$$

It is called a poverty trap.

[Graphical Equilibrium Analysis]

Note the special feature of a poverty trap:

- a sufficiently high positive shock (a big push) would move the economy out of the trap and on a path towards the nice equilibrium k^{**}.
- Implicating a special role for FOREIGN AID.

Do poverty traps exist?

- The discussion is not yet over.
- Yes! say Jeffrey Sachs, Angelina Jolie, Bono.
- Probably not, say some distinguished development economists (Rodrik, Easterly, Kraay) $\rightarrow$ the problem of stagnation in poverty is a deeper, structural one.

We will come back to this issue again.

We will use Diamond later again. It turns out to be particularly useful to analyze generational problems:

- altruism, bequests
- fertility
- savings and longevity
- public pensions vs. private private provision of old-age care
- generational burden of public debt.